SEMI-INFINITE HODGE STRUCTURES AND MIRROR SYMMETRY FOR PROJECTIVE SPACES

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ABSTRACT. We express total set of rational Gromov-Witten invariants of $\mathbb{CP}^n$ via periods of variations of semi-infinite Hodge structure associated with their mirror partners. For this explicit example we give detailed description of general construction of solutions to WDVV-equation from variations of semi-infinite Hodge structures of Calabi-Yau type which was suggested in a proposition from our previous paper ([B2] proposition 6.5).

CONTENTS

1. INTRODUCTION

Motivated by higher-dimensional mirror symmetry we introduced in [B2] quantum periods associated with non-commutative deformations of complex manifolds. When it is just a deformation of complex structure then the corresponding quantum periods specialize to usual periods. In another simplifying situation quantum periods reduce to oscillating integrals. In this paper we look closely at this case and associated variations of semi-infinite Hodge structures. It turns out that these variations give mirror partners to projective spaces. Our principal result is the theorem 5.7 expressing the generating function for total collection of rational Gromov-Witten invariants of $\mathbb{CP}^n$ via periods of variations of semi-infinite Hodge structure of its mirror partner.

Let us consider a pair (X, f) where

$$X = \{x_0 \cdot x_1 \cdot \ldots \cdot x_n = 1\} \subset \mathbb{C}^{n+1},\ f : X \to \mathbb{A}^1_{\mathbb{C}},\ f = x_0 + \ldots + x_n$$

Such data was conjectured (see [G], [EHX]) to be mirror partner of projective space $\mathbb{CP}^n$. Generating function (potential) for $genus = 0$ Gromov-Witten invariants of

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$\mathbb{CP}^n$ reads as (see [KM] eq.(5.15))

(1.1)

$$\mathcal{F}^{\mathbb{CP}^n} = \frac{1}{6}\sum_{0\le i,j,k\le n} y^i y^j y^k \delta_{i+j,n-k} + \sum_{d;m_2,\dots,m_n} N(d;m_2,\dots,m_n)\frac{(y^2)^{m_2}\dots(y^n)^{m_n}}{m_2!\dots m_n!}e^{dy^1}$$

here $y \in H^*(\mathbb{CP}^n,\mathbb{C})$ and the Taylor coefficients $N(d;m_2,\dots,m_n)$ are the virtual numbers of rational curves of degree d in $\mathbb{CP}^n$ intersecting m_a hyperplanes of codimension a.

Let us put

$$\widehat{F}(x;t) = f + \sum_{m=0}^{n} t^m (\sum_{i=0}^{n} x_i)^m \tag{1.2}$$

An oscillating integral $\psi_k = \int_{\Delta_k}\exp(\frac{\widehat{F}(x;t)}{\hbar})(a_0+\hbar a_1+\dots)\frac{dx_0\dots dx_n}{d(x_0\dots x_n)}$ can be expressed using integration by parts via integrals

$$\varphi_k^m(t,\hbar) = \int_{\Delta_k}\exp(\frac{\widehat{F}(x;t)}{\hbar})(\sum_{i=0}^{n} x_i)^m \frac{dx_0\dots dx_n}{d(x_0\dots x_n)},\ \ \varphi_k^m = \hbar\frac{\partial\varphi_k^0(t,\hbar)}{\partial t^m}$$

as linear combination $\psi_k = \sum_{m=0}^{n} v_m\varphi_k^m$ with coefficients $v_m = v_m^{(0)} + \hbar v_m^{(1)} + \dots$. Let us consider an element $\psi(t,\hbar)$ satisfying the following normalization condition:

$$\psi_k(t,\hbar) = \sum_{m=0}^{n} u_m\varphi_k^m(0,\hbar)$$

for some functions $u_m(t,\hbar) = \delta_{m,0} + \hbar^{-1}u_m^{(-1)}(t)+\dots$, $m=0,\dots,n$ (exact meaning of this condition is given in §4). We will show that if one makes a change of parameters $y^m = u_m^{(-1)}(t)$ then Picard-Fuchs equations for oscillating integrals ψ_k take the form

$$\frac{\partial^2\psi_k}{\partial y^i\partial y^j} = \hbar^{-1}\sum_k A_{ij}^m(y)\frac{\partial\psi_k}{\partial y^m} \tag{1.3}$$

Our principal formula can be written then as

$$\forall\, i,j,m,\ \ A_{ij}^{(n-m)}(y) = \partial^3_{ijm}\mathcal{F}^{\mathbb{CP}^n}(y) \tag{1.4}$$

in other words we give integral representation for structure constants of *full* quantum cohomology algebra of $\mathbb{CP}^n$.

In fact, the functions $\psi_k(t,\hbar)$ should be viewed as periods of variation of semi-infinite Hodge structures attached to (X,f). The semi-infinite analog of Hodge filtration associated with a point t of moduli space of deformations of f is given by subspace generated by cohomology classes of oscillating integrals in the space of sections of local system over $\mathbb{C}\setminus\{0\}$ whose fiber over $\hbar$ is the relative cohomology group $H^n(X,\operatorname{Re}\frac{\widehat{F}(x;t)}{\hbar}\to-\infty;\mathbb{C})$. The normalization condition imposed on element $\psi(t)$ from this subspace means that it should belong at the same time to certain covariantly constant affine subspace.

Another aim of this paper is to consider in this explicit case the general relation, which we described in ([B2], proposition 6.5) between variations of semi-infinite Hodge structures and Frobenius manifolds. We construct in particular in §4 a family of solutions to WDVV equations parameterized by open part in isotropic affine grassmanian. Same result holds true for similar pairs (X,f) satisfying homological

condition of Calabi-Yau type, see remark after proposition 4.9. The approach to WDVV-equations described in this paper arose from an attempt to understand relation of the construction of solutions to these equations from ([B1], section 5) with elegant geometric approach to integrable hierarchies from [SW].

Concerning the previously known results we should mention the paper [M] dealing with the case of $\mathbb{CP}^2$ in which after some change of variables the associativity equation $\widetilde{\mathcal{F}}_{y^2y^2y^2} = \widetilde{\mathcal{F}}^2_{y^1y^1y^2} - \widetilde{\mathcal{F}}_{y^1y^1y^1}\widetilde{\mathcal{F}}_{y^1y^2y^2}$ for non-trivial piece of the potential $\mathcal{F}^{\mathbb{CP}^2}$ was identified with differential equation on (multi)sections of pencil of elliptic curves $z^2 = x(x-1)(x-t)$ written in terms of $\int_\infty^{x(t)} \frac{dx}{z}$. Other relevant important result is an integral representation given in ([G]) for structure constants of *small* quantum cohomology algebra $\mathbb{C}[p,q]/p^{n+1} = q$ of $\mathbb{CP}^n$ and also for its T^n-equivariant counterpart. In terms of enumerative geometry of curves in $\mathbb{CP}^n$ Taylor expansion of structure constants of this algebra encode the simplest number $N(1;0,\ldots,0,2) = 1$ which is the number of lines going through two points in $\mathbb{CP}^n$.

2. Moduli space.

In this section we consider meaning of parameter space of deformation (1.2) from the point of view of deformation theory. The content of this section is not used below and readers interested only in explicit formulas giving integral representations for Gromov-Witten invariants may skip it.

Deformations of the pair (X, f) can be described in general with help of differential graded Lie algebra of polyvector fields

$$\mathfrak{g}(X,f) = \oplus_i \mathfrak{g}^i(X,f)[-i], \;\; \mathfrak{g}^i(X,f) := \Gamma_{Zar}(X, \Lambda^{1-i}T), \;\; d := [f,\cdot]$$

equipped with the Schouten bracket. Explicitly, the elements $\gamma \in \mathfrak{g}(X,f)$ can be uniquely written in the form

$$\gamma = \sum_{I\subseteq\{1,\ldots,n\}} \gamma_I(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})\partial_{i_1}\wedge\ldots\wedge\partial_{i_k}$$

where $\gamma_I \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. The space of first order deformations at the base point $T_{[(X,f)]}\mathcal{M}^{(X,F)}$ equals to total cohomology group of complex $(\mathfrak{g}(X,f),[f,\cdot])$. Simple calculation shows that the cohomology of complex $(\mathfrak{g}(X,f),[f,\cdot])$ are zero everywhere except in degree zero where they are $(n+1)-$dimensional. Notice that the partial derivatives $\frac{\partial \widehat{F}}{\partial t_i}|_{t=0}$ $i = 0\ldots n$, where $\widehat{F}(x;t_0,\ldots,t_n)$ is the family of functions defined in (1.2), form a basis in the cohomology of the complex $(\mathfrak{g}(X,f),[f,\cdot])$. It follows that $\mathfrak{g}(X,f)$ is formal as differential graded Lie algebra and that the set $\mathcal{M}^{(X,F)}(R)$ of equivalence classes of solutions to Maurer-Cartan equation in $\mathfrak{g}^1\otimes R$ for an Artin algebra R can be identified with

$$\mathcal{M}^{(X,F)}(R) = \{F = f + \widetilde{f}, \widetilde{f} \in \Gamma_{Zar}(X,\mathcal{O}_X)\otimes\mathfrak{M}_R\}/\{\varphi \in Aut_{Zar}(X)_{/R}, \varphi(0) = Id\}$$

in other words $\mathcal{M}^{(X,F)}$ is the moduli space of deformations of the algebraic function f. It follows that $\widehat{F}(x;t_0,\ldots,t_n)$ is a mini-versal family for such deformations. In the sequel it will be convenient for us to denote by $\mathcal{U}$ the germ of $(n+1)-$dimensional smooth analytic space of the base of the deformation $\widehat{F}(x;t_0,\ldots,t_n)$ and by $\widetilde{X} = X\times\mathcal{U}$ the total space on which $\widehat{F}(x;t_0,\ldots,t_n)$ is defined.

3. Variation of semi-infinite Hodge structures associated with $(X, \widehat{F})$.

In this section we define semi-infinite analog of variation of Hodge structures associated with the moduli space $\mathcal{M}^{(X,F)}$. The analogy with usual variations of Hodge structures associated with families of complex manifolds is explained in [B2] §3.

Let $\mathcal{R}_f$ denotes the sheaf of relative cohomology whose fiber over $\hbar \in \mathbb{C} \setminus \{0\}$ equals to

$$\mathcal{R}_{f,\hbar} := H^n(X, \operatorname{Re}\frac{f}{\hbar} \to -\infty; \mathbb{C}) \tag{3.1}$$

Since there is a natural integral structure on this sheaf: $\mathcal{R}_{f,\hbar} = H^n(X, \operatorname{Re}\frac{f}{\hbar} \to -\infty; \mathbb{Z}) \otimes \mathbb{C}$, therefore it is equipped with Gauss-Manin connection which we will denote by $\mathcal{D}_{\partial/\partial\hbar}$. Denote also via $(\mathcal{R}_{\widehat{F}(t)}, \mathcal{D}^{\mathcal{R}_{\widehat{F}(t)}}_{\partial/\partial\hbar})$ the analogous sheaf with connection associated with the function $\widehat{F}(t)$, $t \in \mathcal{U}$. There is a flat connection $\nabla^{\mathcal{M}}$, $[\nabla^{\mathcal{M}}, \mathcal{D}^{\mathcal{R}_{\widehat{F}(t)}}_{\partial/\partial\hbar}] = 0$ on the total family of sheaves with connections $(\mathcal{R}_{\widehat{F}(t)}, \mathcal{D}^{\mathcal{R}_{\widehat{F}(t)}}_{\partial/\partial\hbar})$, $t \in \mathcal{M}$.

Let us consider subspace of sections of sheaf $\mathcal{R}_{\widehat{F}(t)}$ which are represented by oscillating integrals. In other words we consider the subspace which consists of sections given by integrals of elements of the form $[\exp(\frac{1}{\hbar}\widehat{F}(t))\sum_{i\geq 0}\hbar^i\varphi_i]$, where $\sum_{i\geq 0}\hbar^i\varphi_i \in \Gamma(X \times \mathbb{A}^1_{\mathbb{C},\mathrm{an}}, \Omega^n_{X\times\mathbb{A}^1_{\mathbb{C},\mathrm{an}}/\mathbb{A}^1_{\mathbb{C},\mathrm{an}}})$ (here $\mathbb{A}^1_{\mathbb{C},\mathrm{an}}$ denotes the spectrum of the algebra of analytic functions in $\hbar$).

A natural basis in $H_n(X, \operatorname{Re}\frac{f}{\hbar} \to -\infty; \mathbb{C})$ can be constructed by means of Morse theory. Namely given a critical point p of f and a choice of metric on X the set of all gradient lines $\rho(s)$ of $\operatorname{Re}\frac{f}{\hbar}$ such that $\rho(s) \to p$ as $s \to +\infty$ form relative cycle $\Delta^+_p(\hbar) \in H_n(X, \operatorname{Re}\frac{f}{\hbar} \to -\infty; \mathbb{C})$. One can see easily using standard methods of toric geometry that there exists partial smooth compactification $\overline{X}$ of X such that extension of f on $\overline{X}$ is proper and such that its restricton on $\overline{X} \setminus X$ has no critical points. Then it is easy to see that there exists a metric on $\overline{X}$ such that gradient flow of $\operatorname{Re}(f/\hbar)$ preserves $\overline{X} \setminus X$. Let $\Delta^+_{p_i}(\hbar)$ are relative cycles constructed with help of restriction of this metric on X, where $\{p_i\}$, p_i: $x_0 = \ldots = x_n = \exp\frac{2\pi\sqrt{-1}i}{n+1}$, $i = 0, \ldots, n$ is the set of critical points of f. Standard arguments of Morse theory show that $\{\Delta^+_{p_i}(\hbar)\}$ is a basis in $H_n(X, \operatorname{Re}\frac{f}{\hbar} \to -\infty; \mathbb{C})$.

Let us describe explicitly the subspace of oscillating integrals at $t = 0$. It is convenient to introduce an auxiliary variable α. Let us denote via $\xi_k(\hbar)$ the coefficient in front of α^k in the expression

$$\xi(\alpha, \hbar) := \exp(-\alpha(n+1)\log\hbar)\sum_{d=0}^{\infty}\frac{1}{\hbar^{(n+1)d}[\Gamma(\alpha+d+1)]^{n+1}} \tag{3.2}$$

where $\frac{1}{\Gamma(\alpha+d+1)} = \frac{1}{\Gamma(\alpha+1)}\prod_{i=1}^{d}\frac{1}{i}(1 - \frac{\alpha}{i} + (\frac{\alpha}{i})^2 - \ldots)$

Proposition 3.1.

$$\int_{\Delta_k\subset X} exp\,(\frac{\sum_{i=0}^n x_i}{\hbar})\frac{dx_0\ldots dx_n}{d(x_0\ldots x_n)} = \xi_k(\hbar), \quad k = 0, \ldots, n \tag{3.3}$$

for some locally constant basis $\{\Delta_k(\hbar)\}$ in $H_n(X, Re\,\frac{f}{\hbar} \to -\infty; \mathbb{C})$.

Proof. Recall the integral representation:

$$\frac{1}{\Gamma(s+1)} = \frac{1}{2\pi\sqrt{-1}} \int_C z^{-s} \exp z \frac{dz}{z}$$

where C is the following contour on the complex plane $\{z|Im\, z = 0-,\ Re\, z \in]-\infty, -a]\} \sqcup \{z||z| = a\} \sqcup \{z|Im\, z = 0+,\ Re\, z \in]-\infty, -a]\}. a > 0$. Therefore

$$\xi = \frac{1}{(2\pi\sqrt{-1})^{n+1}} \sum_{d=0}^{\infty} \int \ldots \int_{C\times\ldots\times C} (\hbar z_0 \cdot \ldots \cdot \hbar z_n)^{-d-\alpha} \exp(z_0 + \ldots + z_n) \frac{dz_{0\ldots}dz_n}{z_{0\ldots}z_n}$$

Notice that because $\frac{1}{\Gamma(d+1)} = 0$ for $d < 0$ the first $(n+1)$ Taylor coefficients of $\xi(\alpha, \hbar)$ do not change if one takes $d = -\infty$ as lower limit in summation. Now after making a change of variables $x_1 = \hbar z_1,\ ...,x_n = \hbar z_n,\ y = \prod_{i=0}^{n}(\hbar z_i)$ and noticing that $\sum_d y^{-d} = \delta(y-1)$ in appropriate distributional sense one arrives at 3.3. We leave details to the interested reader. Similar formulas can be found in [GKZ]. Alternatively one can notice that both $\xi_k(\hbar)$ and $\int_{\Delta^+_{p_i}} e^{f/\hbar} dx_0 \ldots dx_n / d(x_0 \ldots x_n)$ satisfy $(n+1)-$st order differential equation $(-\frac{\hbar}{n+1}\frac{\partial}{\partial \hbar})^{n+1} s = \hbar^{-(n+1)} s$. It follows from expansions

$$\int_{\Delta^+_{p_i}} e^{\frac{f}{\hbar}} \frac{dx_0 \ldots dx_n}{d(x_0 \ldots x_n)} = e^{\frac{f(p_i)}{\hbar}} (\mathrm{const}\, \hbar^{\frac{n}{2}} + O(\hbar^{\frac{n}{2}+1})) \text{ as } \hbar \to 0$$

and $\xi_k(\hbar) = (-(n+1)\log \hbar)^k + O((\log \hbar)^{k-1})$ as $\hbar \to \infty$ that we have two $(n+1)$ -tuples of linearly independent solutions, which therefore must be related by a linear transformation. □

Consequently the subspace of sections of sheaf $\mathcal{R}_f$ represented by oscillating integrals can be described explicitly as subspace generated by elements which in the locally constant frame dual to $\{\Delta_k\}$ are written as

$$\hbar^{2l+m} (\frac{\partial^l \xi_k(\hbar)}{\partial \hbar^l})_{k=0,\ldots,n},\ l \in \{0, \ldots, n\},\ m \in \mathbb{N} \cup \{0\}$$

It is convenient to introduce the algebra $\mathbb{C}[\alpha]/\alpha^{n+1}\mathbb{C}[\alpha]$ and to identify $\{\alpha^k\}_{k=0,\ldots,n}$ with locally constant frame dual to $\{\Delta_k\}$.Then monodromy transformation around $\hbar = 0$ acting on cohomology $H^n(X, \mathrm{Re}\, \frac{f}{\hbar} \to -\infty; \mathbb{C})$ is given in the basis $\{\alpha^k\}$ by multiplication by $\exp((n+1)2\pi\sqrt{-1}\alpha)$. Notice that the set $\{\hbar^{-\alpha(n+1)}\alpha^k\}$ is a single-valued frame in $\mathcal{R}_f$. Below α will always denote an element satisfying $\alpha^{n+1} = 0$.

Let us consider a completion of the space $\Gamma(\mathbb{A}^1_{\mathbb{C},\mathrm{an}} \setminus \{0\}, \mathcal{R}_f)$ defined as Hilbert space $H = \Gamma_{L^2}(S^1, \mathcal{R}_f|_{S^1})$ consisting of L^2-sections of restriction of the sheaf $\mathcal{R}_f$ to the circle $S^1 = \{\hbar | \hbar \in \mathbb{C},\ |\hbar| = R,\ R > 0\}$. Explicitly, elements of H can be written as

$$\hbar^{-\alpha(n+1)} \sum_{i=0}^{n} \alpha^i \zeta_i(\hbar), \text{ where } \zeta_i(\hbar) \in L^2(S^1, \mathbb{C})$$

where $\hbar^{-\alpha(n+1)} := \exp(-\alpha(n+1)\log \hbar)$. Let us consider Segal-Wilson grassmanian Gr associated with H. We introduce polarization $H = H^+ \oplus H^-$ where H^+ and H^- are the closed subspaces generated by elements of the form $\hbar^{-\alpha(n+1)+k}\alpha^i$ for $k \geq 0$ and $k < 0$ respectively. Recall (see [PS]) that Segal-Wilson grassmanian consists of closed subspaces $L \subset H$ "comparable" with H^+: $pr_+ : L \to H^+$ is a Fredholm operator and $pr_- : L \to H^-$ is a Hilbert-Schmidt operator.

The semi-infinite subspace $L(t)$ associated with a point $t \in \mathcal{U}$ is defined as closure of the subspace of oscillating integrals, i.e. it is the closed subspace generated by elements , which in the frame dual to $\{\Delta_k\}$ have coordinates

$$\int_{\Delta_k(t,\hbar)} \exp(\frac{\widehat{F}(x;t)}{\hbar})\,\varphi\,\hbar^i,\;\; i \geq 0,\;\; \varphi \in \Gamma_{Zar}(X, \Omega_X^{n+1}) \tag{3.4}$$

where $\Delta_k(t,\hbar)$, is covariantly constant family of elements in $H^n(X, \mathrm{Re}\,\frac{\widehat{F}(x;t)}{\hbar} \to -\infty; \mathbb{C})$ such that $\Delta_k(0,\hbar)$ coincides with $\Delta_k(\hbar)$ from prop.3.1. More precisely we consider $(n+1)\times(n+1)$ matrix ϑ whose entries are series of hypergeometric type obtained via expansion $\exp(\sum_{m=0}^n t^m f^m/\hbar) = \sum_l s^l(t,\hbar) f^l$ and termwise integration:

$$\vartheta_{kj} = \int_{\Delta_k(t)} \exp(\frac{f + \sum_{m=0}^n t^m f^m}{\hbar}) f^j \frac{dx_0 \ldots dx_n}{d(x_0 \ldots x_n)} = \sum_{l=j}^{\infty} s^{l-j}(t,\hbar) \int_{\Delta_k} e^{\frac{f}{\hbar}} f^l \frac{dx_0 \ldots dx_n}{d(x_0 \ldots x_n)}$$

where $\int_{\Delta_k} e^{\frac{f}{\hbar}} f^l \frac{dx_0 \ldots dx_n}{d(x_0 \ldots x_n)} = (\frac{\partial}{\partial(\hbar^{-1})})^l \xi_k$ and $\xi_k(\hbar)$ is defined in (3.2). The resulting series is convergent for $|\hbar| > C$ for some constant $C > 0$ and small t. We set $L(t) := \vartheta H^+$. It follows from [PS] prop. 6.3.1 that $L(t) \in Gr$.

The family of subspaces $L(t)$ has the following basic property.

Proposition 3.2. *The family of subspaces $L(t), t \in \mathcal{U}$ satisfies semi-infinite analog of Griffiths transversality condition with respect to the Gauss-Manin connection $\nabla^{\mathcal{M}}$:*

$$\forall\, i \;\; \frac{\partial}{\partial t^i} L(t) \subseteq \hbar^{-1} L(t) \tag{3.5}$$

Proof. It follows from $\frac{\partial}{\partial t^i}\exp(\frac{\widehat{F}(t)}{\hbar}) = \frac{1}{\hbar} f^i \exp(\frac{\widehat{F}(t)}{\hbar})$ □

The induced map (symbol of the Gauss-Manin connection):

$$Symbol(\nabla^{\mathcal{M}}) : L([f])/(\hbar L([f])) \otimes T_{[f]}\mathcal{M} \to (\hbar^{-1} L([f]))/L([f])$$

is easy to describe. At $[f] \in \mathcal{M}$ space $L([f])/(\hbar L([f])) \simeq (\hbar^{-1} L([f]))/L([f])$ is identified naturally with

$$\Gamma_{Zar}(X, \Omega_X^n)/\{d_X f \wedge \nu \,|\, \nu \in \Gamma_{Zar}(X, \Omega_X^{n-1})\}$$

and

$$T_{[f]}\mathcal{M} = \Gamma_{Zar}(X, \mathcal{O}_X)/\{Lie_v f) \,|\, v \in \Gamma_{Zar}(X, \mathcal{T}_X)\} \tag{3.6}$$

Then

$$Symbol(\nabla^{\mathcal{M}})(\varphi \otimes \mu) = [\varphi \cdot \mu]$$

4. Solutions to WDVV-equation.

In this section we will show how to construct a solution to WDVV-equations associated with (X, f) given an arbitrary element from an open domain in isotropic affine grassmanian of H.

In the next section we single out a solution which coincides with solution defined by Gromov-Witten invariants of $\mathbb{CP}^n$.

4.1. **Picard-Fuchs equation for periods of the semi-infinite Hodge structure.** We start from fixing a choice of semi-infinite subspace $S \subset H$, $\hbar^{-1}S \subset S$ transversal to $L(0)$:

$$H = L(0) \oplus S \tag{4.1}$$

S belongs to the same type of grassmanian related with polarization where roles of H^+ and H^- are inversed.

Next we would like to choose an element $\Omega_0 \in L(0)$ such that the symbol of Gauss-Manin connection restricted to the class $[\Omega_0]\, mod\, (\hbar L([f]))$ in the quotient $L([f])/(\hbar\, L([f]))$ gives an isomorphism

$$Symbol(\nabla^{\mathcal{M}})([\Omega_0] \otimes \cdot) : T_{[f]}\mathcal{M} \to (\hbar^{-1}L([f]))/L([f]) \tag{4.2}$$

It is easy to see that

$$T_{[f]}\mathcal{M} \simeq \mathbb{C}[p]/(p^{n+1}-1)\,\text{where}\, p = [(\frac{1}{n+1}\sum_{i=0}^{n} x_i)]$$

$$L([f])/(\hbar L([f])) \simeq \left(\mathbb{C}[p]/(p^{n+1}-1)\right)[\frac{dx_{0...}dx_n}{d(x_{0...}x_n)}]$$

So one can take here as Ω_0 for example an arbitrary element of the form

$$\left[\exp(\frac{1}{\hbar}f)\left((\sum_{i=0}^{n} x_i)^k + \hbar\omega_1 + \dots\right)\frac{dx_{0...}dx_n}{d(x_{0...}x_n)}\right]\; k = 0, \dots, n$$

The transversality condition (4.1) implies that the intersection of $L(t)$ for $t \in \mathcal{U}$ with affine space $\{\Omega_0 + \eta | \eta \in S\}$ consists of a single element. Let us denote it via Ψ^S:

$$\{\Psi^S(t,\hbar)\} = L(t) \cap \{\Omega_0 + \eta | \eta \in S\}$$

Notice that the property (4.2) and the mini-versality of the family $\widehat{F}(t)$ implies that $\{\partial_i \Psi^S(t) \in \hbar^{-1}L(t) \bmod L(t)\}$ is a basis in $(n+1)-$dimensional vector space $(\hbar^{-1}L(t))/L(t)$ and therefore $\{\partial_i \Psi^S(t) \in S \bmod \hbar^{-1}S\}$ is a basis in $S/(\hbar^{-1}S)$. Hence the map

$$\Psi^S(t,\infty) := [\Psi^S(t,\hbar) - \Omega_0] \in S \bmod \hbar^{-1} S \tag{4.3}$$

is a local isomorphism and induces a set of coordinates on $\mathcal{U}$ which we denote by $\{t_S^a\}$. Let us denote via $\Psi^{S,k} = \int_{\Delta_k(t_S^a,\hbar)} \Psi^S$ the components of Ψ^S with respect to the basis α^k.

Proposition 4.1. *The periods $\Psi^{S,k}(t_S^a,\hbar)$ satisfy*

$$\frac{\partial^2 \Psi^{S,k}}{\partial t_S^a \partial t_S^b} = \hbar^{-1}\sum_c A_{ab}^c(t_S)\frac{\partial \Psi^{S,k}}{\partial t_S^c} \tag{4.4}$$

Proof. It follows from the prop. 3.5 that

$$\frac{\partial^2 \Psi^S}{\partial t^i \partial t^j} \in \hbar^{-2}L(t)$$

Therefore one has

$$\frac{\partial^2 \Psi^S}{\partial t^i \partial t^j} - \hbar^{-1}\sum_m \left(A^{(-1)}\right)_{ij}^m(t)\frac{\partial \Psi^S}{\partial t^m} - \sum_m \left(A^{(0)}\right)_{ij}^m(t)\frac{\partial \Psi^S}{\partial t^m} \in L(t)$$

for some $\left(A^{(-1)}\right)_{ij}^{m}(t)$, $\left(A^{(0)}\right)_{ij}^{m}(t) \in \mathcal{O}_{\mathbb{C}}^{analytic}$. On the other hand,

$$\frac{\partial^2 \Psi^S}{\partial t^i \partial t^j}, \frac{\partial \Psi^S}{\partial t^i} \in S$$

Therefore

$$\frac{\partial^2 \Psi^S}{\partial t^i \partial t^j} = \hbar^{-1} \sum_m \left(A^{(-1)}\right)_{ij}^{m}(t) \frac{\partial \Psi^S}{\partial t^m} + \sum_m \left(A^{(0)}\right)_{ij}^{m}(t) \frac{\partial \Psi^S}{\partial t^m}$$

The coordinates $\{t_S^a\}$ were chosen so that $\Psi^S \bmod \hbar^{-1} S : \mathcal{U} \to S/\hbar^{-1}S$ is linear in $\{t_S^a\}$. Therefore in these coordinates $\left(A^{(0)}\right)_{ab}^{c} = 0$. $\square$

Corollary 4.2. *One has* $[A, A] = 0$, $dA = 0$ *for* $A = \sum_a A_{ab}^c(t_S) dt_S^a$.

In particular, the following formula defines commutative and associative product "$\circ$" on tangent sheaf of $\mathcal{U}$:

$$\frac{\partial}{\partial t^a} \circ \frac{\partial}{\partial t^b} := \sum_c A_{ab}^c(t_S) \frac{\partial}{\partial t^c} \tag{4.5}$$

4.2. **Flat metrics.** Next ingredient needed for construction of the solution to WDVV-equation is a flat metric on $\mathcal{U}$ compatible with multiplication (4.5). In this subsection we show that the period map Ψ^S induces such a metric under condition that the semi-infinite subspace S is isotropic with respect to certain pairing.

Let us notice that sheaf $\mathcal{R}_{\widehat{F}(t)}$ is equipped with natural pairing between the opposite fibers:

$$G(\hbar): \ H^n(X, \operatorname{Re} \frac{\widehat{F}(t)}{\hbar} \to -\infty; \mathbb{C}) \otimes H^n(X, \operatorname{Re} - \frac{\widehat{F}(t)}{\hbar} \to -\infty; \mathbb{C}) \to \mathbb{C}$$

which is given by the Poincare pairing dual to intersection pairing on relative cycles. Hence by continuity one has natural pairing on elements of $L(t)$, $t \in \mathcal{U}$.

Lemma 4.3. *For* $\mu(\hbar), \eta(\hbar) \in L(t) \subset H$

$$G(\mu, \eta) \in \hbar^n \mathbb{C}[[\hbar]]$$

Proof. One has for $\hbar$ from any sector $a < \arg(\hbar) < b$, $b - a < \pi$ and $\mu(\hbar), \eta(\hbar) \in L(t)$:

$$G(\mu, \eta) = \sum_{i,j} \#(\Delta_i^+ \cap \Delta_j^-) (\int_{\Delta_i^+} \mu(\hbar)) (\int_{\Delta_j^-} \eta(-\hbar))$$

where sets $\{\Delta_i^+\}$,$\{\Delta_j^-\}$ are frames of relative cycles in $H_n(X, \operatorname{Re} \frac{\widehat{F}(t)}{\hbar} \to -\infty; \mathbb{C})$, $H_n(X, \operatorname{Re} - \frac{\widehat{F}(t)}{\hbar} \to -\infty; \mathbb{C})$ correspondingly. Expansion for the integrals $\int_{\Delta_i^+} \mu(\hbar))$, $\int_{\Delta_j^-} \eta(-\hbar)$ at $\hbar \to 0$ can be evaluated using the steepest-descent method. Recall that given a non-degenerate critical point p of $\widehat{F}(t)$ and a metric one has two relative cycles Δ_p^+, Δ_p^- from $H_n(X, \operatorname{Re} \frac{\widehat{F}(t)}{\hbar} \to -\infty; \mathbb{C})$ and $H_n(X, \operatorname{Re} - \frac{\widehat{F}(t)}{\hbar} \to -\infty; \mathbb{C})$ respectively which are formed by gradient lines $\rho(r)$ of $\operatorname{Re} \widehat{F}(t)/\hbar$ such that $\rho(r) \to p$ as $r \to +\infty$ and $\rho(r) \to p$ as $r \to -\infty$ respectively. Then one has the following asymptotic expansions

$$\int_{\Delta_p^+} e^{\frac{\widehat{F}(x;t)}{\hbar}} (\varphi_0 + \hbar \varphi_1 + \dots) = e^{\frac{\widehat{F}(p;t)}{\hbar}} (\mathrm{const}^{(+)} \, \hbar^{\frac{n}{2}} + O(\hbar^{\frac{n}{2}+1})) \tag{4.6}$$

$$\int_{\Delta_p^-} e^{-\frac{\widehat{F}(x;t)}{\hbar}}(\varphi_0 - \hbar\varphi_1 + \dots) = e^{-\frac{\widehat{F}(p;t)}{\hbar}}(\mathrm{const}^{(-)}\,\hbar^{\frac{n}{2}} + O(\hbar^{\frac{n}{2}+1}))$$

Recall that f has $(n+1)$ non-degenerate critical points p_m, p_m: $x_0 = \dots = x_n = \exp\frac{2\pi\sqrt{-1}m}{n+1}$ and that the relative cycles $\{\Delta^+_{p_m}\}$, (resp. $\{\Delta^-_{p_m}\}$) form basis in $H_n(X, \mathrm{Re}\,\frac{f}{\hbar} \to -\infty; \mathbb{C})$, (resp. $H_n(X, \mathrm{Re} - \frac{f}{\hbar} \to -\infty; \mathbb{C})$). It follows that

$$G(\mu, \eta) = \mathrm{const}\,\hbar^n + O(\hbar^{n+1})$$

□

Let us assume now that S is isotropic with respect to G in the following sense:

$$\forall\ \mu(\hbar), \eta(\hbar) \in S\ \ G(\mu, \eta) \in \hbar^{n-2}\mathbb{C}[[\hbar^{-1}]] \tag{4.7}$$

An example of such subspace is considered in the next section. It is the subspace generated by elements $(\hbar\alpha)^k\hbar^{-m-(n+1)\alpha}$, $k \in \{0, \dots, n\}$, $m \in \mathbb{N}$.

The period map Ψ^S induces a pairing on $\mathcal{TM}$:

$$\frac{\partial}{\partial t^a} \otimes \frac{\partial}{\partial t^b} \to G(\frac{\partial \Psi^S}{\partial t^a}, \frac{\partial \Psi^S}{\partial t^b}) \tag{4.8}$$

Proposition 4.4. *For isotropic S the pairing induced by Ψ^S satisfies*

$$G(\frac{\partial \Psi^S}{\partial t^a}, \frac{\partial \Psi^S}{\partial t^b}) = \hbar^{n-2} g_{ab} \tag{4.9}$$

with $g_{ab}(t)$ symmetric and non-degenerate. In the coordinates $\{t^a_S\}$ induced by the map $\Psi^S(t, \hbar = \infty)$ the 2-tensor g_{ab} is constant.

Proof. The property (4.9) follows from $\partial_a\Psi^S(t, \hbar) \in S \cap \hbar^{-1}L(t)$. Notice that for any $\mu(\hbar), \eta(\hbar) \in H$:

$$G(\mu, \eta)(\hbar) = (-1)^n G(\eta, \mu)(-\hbar)$$

It follows that g_{ab} is symmetric. Non-degeneracy of g_{ab} follows from the analogous property of G.

Notice that G is obviously $\mathbb{C}[\hbar^{-1}]-$linear and therefore one has induced pairing G_0 with values in $\mathbb{C} \cdot \hbar^{n-2}$ on elements of $S/\hbar^{-1}S$. It follows from (4.9) that

$$G(\frac{\partial \Psi^S}{\partial t^a_S}, \frac{\partial \Psi^S)}{\partial t^b_S}) = G_0([\frac{\partial \Psi^S}{\partial t^a_S}], [\frac{\partial \Psi^S}{\partial t^b_S}])$$

By definition of the coordinates $\{t^a_S\}$ one has $[\frac{\partial \Psi^S}{\partial t^a_S}] =$ *constant* as an element of $S/\hbar^{-1}S$. □

Proposition 4.5. *Metric g_{ab} is compatible with multiplication defined by $A^c_{ab}(t_S)$:*

$$\forall\, a, b, c \quad g(a \circ b, c) = g(a, b \circ c) \tag{4.10}$$

Proof. Notice that G is locally constant with respect to the Gauss-Manin connection $\nabla^{\mathcal{M}}$. Therefore

$$\partial_b G(\partial_a\Psi, \partial_c\Psi) = G(\partial_b\partial_a\Psi, \partial_c\Psi) + G(\partial_a\Psi, \partial_b\partial_c\Psi)$$

In the coordinates $\{t^a_S\}$ one has

$$\partial_b G(\partial_a\Psi, \partial_c\Psi) = 0,\ \ \partial_a\partial_b\Psi = \hbar^{-1}\sum_d A^d_{ab}(t_S)\partial_d\Psi$$

The pairing G satisfies $G(\hbar^{-1}\mu,\eta) = -G(\mu,\hbar^{-1}\eta) = \hbar^{-1}G(\mu,\eta)$ □

Let us denote

$$A_{abc}(t_S) := g(\partial_a \circ \partial_b, \partial_c)$$

Corollary 4.2 and proposition 4.5 imply now that

$$A_{abc}(t_S) = \partial_a\partial_b\partial_c\Phi^S(t_S)$$

for some function Φ^S defined up to addition of terms of order ≤ 2, and that the function $\Phi^S(t_S)$ satisfy the system of WDVV-equations:

$$\forall\ a,b,c,d \quad \sum_{e,f}\partial_a\partial_b\partial_e\Phi^S g^{ef}\partial_f\partial_c\partial_d\Phi^S = \sum_{e,f}\partial_a\partial_d\partial_e\Phi^S g^{ef}\partial_f\partial_b\partial_c\Phi^S \tag{4.11}$$

4.3. **Symmetry vector field.** Let us notice that the subspace $L(t)$ satisfies

$$\mathcal{D}_{\partial/\partial\hbar}\, L(t) \subseteq \hbar^{-2}L(t)$$

Assume now that the subspace $S \subset H$ satisfies the condition

$$\mathcal{D}_{\partial/\partial\hbar}\, S \subseteq \hbar^{-1}S \tag{4.12}$$

Notice that this condition is enough to check only on finite set of elements from S which form a basis of $S/\hbar^{-1}S$. Assume that one has also $\mathcal{D}_{\partial/\partial\hbar}\,\Omega_0 \in \hbar^{-1}S$.

Proposition 4.6. *The periods $\Psi^{S,k}(t^S)$ satisfy*

$$\frac{\partial\Psi^{S,k}}{\partial\hbar} = \hbar^{-1}E^a(t_S)\frac{\partial\Psi^{S,k}}{\partial t_S^a} \tag{4.13}$$

for some vector field $E = \sum_a E^a(t^S)\partial_a$

Proof. It follows from $\frac{\partial\Psi^{S,k}}{\partial\hbar} \in \hbar^{-2}L(t) \cap \hbar^{-1}S$. The arguments are similar to the arguments from the proof of the proposition 4.1. □

Proposition 4.7. *The vector field E acts as a conformal symmetry on the multiplication $\circ$ and the metric g_{ab}, that is, for any vector fields $u, v \in \mathcal{T}_{\mathcal{M}}$*

$$[E, u\circ v] - [E,u]\circ v - u\circ[E,v] = u\circ v \tag{4.14}$$

$$Lie_E g(u,v) - g([E,u],v) - g(u,[E,v]) = (2-n)g(u,v) \tag{4.15}$$

Proof. It follows from $[\nabla^{\mathcal{M}}, \mathcal{D}^{\mathcal{R}_{\hat{F}(t)}}_{\partial/\partial\hbar}] = 0$ and $\mathcal{D}^{\mathcal{R}_{\hat{F}(t)}}_{\partial/\partial\hbar}G = 0$ respectively. □

4.4. **Flat identity vector field.** Assume that the subspace S satisfies the condition

$$\hbar^{-1}\Omega_0 \in S \tag{4.16}$$

Let e denotes vector field on $\mathcal{M}$ which is affine in the coordinates $\{t_S\}$ and which corresponds to the class of $\hbar^{-1}\Omega_0$ in the quotient $S/\hbar^{-1}S$. We can assume that $e = \frac{\partial}{\partial t_S^0}$ where t_S^0 is one of the affine coordinates.

Proposition 4.8. *The vector field $e = \frac{\partial}{\partial t_S^0}$ is the identity with respect to the multiplication $\circ$.*

Proof. One has $\hbar^{-1}\Psi^S \in \hbar^{-1}L(t)$, $\hbar^{-1}\Psi^S \in S$. Therefore

$$\hbar^{-1}\Psi^S(t_S) = \frac{\partial \Psi^S(t_S)}{\partial t_S^0}$$

Differentiating this equality with respect to $\frac{\partial}{\partial t_S^a}$ and using eq.(4.4) we see that $\frac{\partial}{\partial t_S^a} \circ \frac{\partial}{\partial t_S^0} = \frac{\partial}{\partial t_S^a}$. □

In summary, we have

Proposition 4.9. *For any semi-infinite subspace S satisfying the properties (4.1), (4.7), (4.12), (4.16) we have constructed Frobenius manifold structure on $\mathcal{U}$.*

For the definition of Frobenius structure see [D],[KM].

Remark 4.10. Similarly, there exists a family of solutions to WDVV-equations parameterized by an open domain in isotropic affine grassmanian of semi-infinite subspaces given an arbitrary similar pair (X, f) satisfying the following homological Calabi-Yau type condition: there exists $\Omega_0 \in \Gamma_{Zar}(X, \Omega^n)$, $n = \dim_{\mathbb{C}} X$ such that the restriction on $[e^{\frac{f}{\hbar}}\Omega_0]$ of the symbol of Gauss-Manin connection induces isomorphism $T_{[f]}\mathcal{M} \simeq (\hbar^{-1}L([f]))/L([f])$, here under $T_{[f]}\mathcal{M}$ we understand the corresponding space of first-order deformations i.e. the vector space $\Gamma_{Zar}(X, \mathcal{O}_X)/\{Lie_v f \,|\, v \in \Gamma_{Zar}(X, \mathcal{T}_X)\}$. In particular this is true under some mild conditions whenever X admits a nowhere vanishing "holomorphic" volume element. Also similar statement, which deals with total sum of cohomology of complex of sheaves $(\Lambda^* TX_{Zar}, \{f, \cdot\})$ and subspace of exponential integrals in total cohomology $H^*(X, \mathrm{Re}\,\frac{f}{\hbar} \to -\infty; \mathbb{C})$, holds true. We assume in the previous statement unobstructedness of the corresponding moduli space, which becomes non-trivial in such more general setting as deformations become more of a noncommutative flavour.

Remark 4.11. From some point of view the construction mentioned in the first part of previous remark gives a certain global analogue of K. Saito and M.Saito theory ([S,S]). This theory provides a construction of a Frobenius manifold starting from a germ of holomorphic function having *one* isolated singularity. The construction uses microlocal algebraic analysis and, in particular, solution to a certain Riemann-Hilbert problem obtained from some deep results from theory of D-modules. On one hand in some cases, like quasi-homogenious singularity, the local data involving germ of isolated singularity can be identified with some global data associated with an algebraic function $\mathbb{A}^n \to \mathbb{A}^1$ of the type mentioned in the previous remark. It can be shown that any Frobenius manifold arises from some abstract semi-infinite variation of Hodge structure $L(t)$ provided with some choice of constant opposite semi-infinite subspace S. Thus it is plausible that in such cases when global and local data coincide and for specific choices of semi-infinite subspace S the construction mentioned in the previous remark reproduces solutions to WDVV-equations which were found in ([S,S]). Approach from previous remark can help to clarify nature of solutions from ([S,S]). In a sense semi-infinite geometry gives here an intrinsic tool for solution of a class of Riemann-Hilbert problems. It should be noted also that solutions to WDVV-equations found in ([S,S]) are difficult to access for explicit calculations. Use of semi-infinite geometry helps in principle to overcome this problem. On the other hand, applying construction from the first part of the previus remark to general global data when, in particular, f has several singularities

one can try to see how the Saito's constructions associated with every singularity of f get "glued" together somehow. Use of semi-infinite geometry could be helpful in clarifying this gluing phenomena and finding global analogue [1] of Saito's theory in most general setting of arbitrary pairs (X, f) [2], which could in turn provide a nice simplest example for study of general semi-infinite variations of Hodge structures of Calabi-Yau type.

5. Rational Gromov-Witten invariants of $\mathbb{CP}^n$ and oscillating integrals.

Consider the subspace $S_0 \subset H$ generated by elements

$$(\hbar\alpha)^k \hbar^{-m-(n+1)\alpha}, \;\; k \in \{0, \dots, n\}, \;\; m \in \mathbb{N}$$

Let us also put $\Omega_0 = [e^{\frac{f}{\hbar}} \frac{dx_0 \dots dx_n}{d(x_0 \dots x_n)}] \in L_0$. In this section we show that for $S = S_0$ the quasi-homogeneous solution to WDVV equation $\partial^3 \Phi^{S_0}(t_{S_0})$ constructed in the previous section coincides with the generating function of Gromov-Witten invariants of $\mathbb{CP}^n$. The idea is to identify some numerical invariants of these two solutions and then use reconstruction theorem - I from [KM].

Proposition 5.1. *The subspace S_0 has the properties (4.1), (4.7), (4.12), (4.16).*

Proof. Properties (4.12) (4.16) are easy to check. In order to check the property (4.1) let us notice that the subspace $L(0)$ is generated by elements

$$\hbar^{2k+m} \frac{\partial^k \xi}{\partial \hbar^k} \operatorname{mod} \alpha^{n+1} \;\; k \in \{0, \dots, n\}, \; m \in \mathbb{N} \cup \{0\}$$

and that

$$\hbar^{(n+1)\alpha} \frac{\partial^k \xi}{\partial \hbar^k} \operatorname{mod} \alpha^{n+1} = \operatorname{const} \hbar^{-k} \alpha^k + O(\hbar^{-(k+1)})$$

To check the property (4.7) let us notice that since $\{\alpha^k\}$ is dual to locally constant basis $\{\Delta_k\}$ and $\alpha^i \hbar^{-(n+1)\alpha}$ are single-valued sections therefore for any i, j the pairing $G\big(\alpha^i \hbar^{-(n+1)\alpha}, \alpha^j \hbar^{-(n+1)\alpha}\big)$ does not depend on $\hbar$:

$$\forall \;\; i, j \;\; G\big(\alpha^i \exp(-(n+1)\log(\hbar)\alpha), \alpha^j \exp(-(n+1)\log(\hbar)\alpha)\big) = \operatorname{const}$$

Differentiating this equality with respect to $\frac{\partial}{\partial \hbar}$ one gets

$$G\big(\alpha^{i+1} \hbar^{-(n+1)\alpha}, \alpha^j \hbar^{-(n+1)\alpha}\big) = G\big(\alpha^i \hbar^{-(n+1)\alpha}, \alpha^{j+1} \hbar^{-(n+1)\alpha}\big)$$

Therefore $G\big(\alpha^i \hbar^{-(n+1)\alpha}, \alpha^j \hbar^{-(n+1)\alpha}\big) = 0$ for $i + j > n$. □

[1] Based on example of quasi-homogenious polynomial existence of some global analogue of Saito's theory was conjectured also in [G]. Reader should be warned that conjecture from [G] must be corrected as the above examples show. Namely, "volume form" ω_λ (in notations of [G]) with prescribed properties does not exist in general, instead there should exist a family of top degree forms $\omega_\lambda^{(0)} + \hbar \omega_\lambda^{(1)} + \dots$ depending on the parameter $\hbar$.

[2] *Note added:* After appearance of present paper C.Sabbah, to whom I would like to express my gratitude for this interesting remark, told me about his recent work ([Sab]) containing extension of parts of Saito's theory in certain global cases. However, global analogue of part of Saito's theory, which involves proof of existence of "primitive form" in families depending on parameters from moduli spaces like $\mathcal{M}^{(X,F)}$ above and consequently gives solutions to WDVV-equations, does not exist in literature to my knowledge.

Let us consider system of local coordinates $\{t^k_{S_0}\}$, $k = 0 \dots n$, induced by the map (4.3) from linear coordinates on $S/\hbar^{-1}S$ corresponding to the basis given by classes of elements $\alpha^k \hbar^{k-1-(n+1)\alpha}$ modulo $\hbar^{-1}S$. Notice that this notation is compatible with notation for the flat identity from §4.4 since $[\hbar^{-1}\Omega_0] = [\hbar^{-1-(n+1)\alpha}]$. We put below $y^k = t^k_{S_0}$ to simplify notations.

Lemma 5.2. *In coordinates y^k the symmetry vector field is written as*

$$E(y^k) = (\sum_{k=0}^{n} (k-1) y^k \frac{\partial}{\partial y^k}) - (n+1)\frac{\partial}{\partial y^1}$$

Proof. One has

$$\Psi^{S_0}(y) = \Omega_0 + \sum_{k=0}^{n} \hbar^{k-1-(n+1)\alpha} \alpha^k y^k \ \ mod \ \hbar^{-1} S_0$$

Therefore

$$\frac{\partial \Psi^{S_0}(y)}{\partial \hbar} =$$

$$= -(n+1)\alpha\hbar^{-1-(n+1)\alpha} + \sum_{k=0}^{n} \hbar^{k-2-(n+1)\alpha}\alpha^k (k-1) y^k \ \ mod \ \hbar^{-2} S_0$$

$$\frac{\partial \Psi^{S_0}(y)}{\partial y^k} = \hbar^{k-1-(n+1)\alpha}\alpha^k \ \ mod \ \hbar^{-1} S_0$$

and

$$\frac{\partial \Psi^{S_0}(y)}{\partial \hbar} = \hbar^{-1}\big(-(n+1)\frac{\partial \Psi^{S_0}(y)}{\partial y^1} + \sum_{k=0}^{n}(k-1)y^k \frac{\partial \Psi^{S_0}(y)}{\partial y^k}\big) \ \ mod \ \hbar^{-2} S_0$$

This is an exact equality by proposition 4.6 □

Lemma 5.3.

$$\frac{\partial \Psi^{S_0,i}(y)}{\partial y^k}|_{y=0} = \frac{1}{\hbar (n+1)^k} \int_{\Delta_i} f^k \exp(\frac{1}{\hbar} f) \frac{dx_{0\dots}dx_n}{d(x_{0\dots}x_n)} \ mod \ \hbar L(0)$$

In other words at $y = 0$ the elements $\frac{\partial}{\partial y^k}$ correspond to elements $(\frac{\sum_{i=0}^{n} x_i}{n+1})^k \in \Gamma_{Zar}(X, \mathcal{O}_X)/\{Lie_v(f) \,|\, v \in \Gamma_{Zar}(X, \mathcal{T}_X)\}$ from $T_{[f]}\mathcal{M}^{(X,f)}$

Proof. One has

$$\hbar^{k-1}(\hbar\frac{\partial}{\partial \hbar})^k \Psi^{S_0}|_{y=0} = \hbar^{k-1}(-(n+1)\alpha)^k \hbar^{-(n+1)\alpha} \ mod \ \hbar^{-1} S_0 =$$

$$= (-(n+1))^k \frac{\partial \Psi^{S_0}(y)}{\partial y^k}|_{y=0} \ mod \ \hbar^{-1} S_0$$

we see that

$$\hbar^{k-1}(\hbar\frac{\partial}{\partial \hbar})^k \Psi^{S_0}|_{y=0} - (-(n+1))^k \frac{\partial \Psi^{S_0}(y)}{\partial y^k}|_{y=0} \in \hbar^{-1} L(0) \cap \hbar^{-1} S_0$$

Therefore $\hbar^{k-1}(\hbar\frac{\partial}{\partial\hbar})^k\Psi^{S_0}|_{y=0} = (-(n+1))^k\frac{\partial\Psi^{S_0}(y)}{\partial y^k}|_{y=0}$ and

$$\frac{\partial\Psi^{S_0,i}(y)}{\partial y^k}|_{y=0} = \frac{1}{\hbar(n+1)^k}\int_{\Delta_i} f^k\exp(\frac{1}{\hbar}f)\frac{dx_0...dx_n}{d(x_0...x_n)}\ mod\ \hbar L(0)$$

□

Corollary 5.4. *At $y=0$ the multiplication (4.5) is given by*

$$A^k_{ij}(0) = \delta_{i+j-k\, mod\, n+1,0}$$

in the basis $\{\frac{\partial}{\partial y^k}\}$.

Proof. $\Gamma_{Zar}(X,\mathcal{O}_X)/\{Lie_v(f)\,|\,v\in\Gamma_{Zar}(X,\mathcal{T}_X)\}\simeq\mathbb{C}[p]/(p^{n+1}-1)$ with $p=[\frac{\sum_{i=0}^n x_i}{n+1}]$. □

Let us calculate now the pairing (4.8) written in coordinates y^k. By proposition 4.4 and lemma 5.3 the value of $\hbar^{-n+2}G(\frac{\partial\Psi^{S_0}}{\partial y^i},\frac{\partial\Psi^{S_0}}{\partial y^j})$ is equal to the coefficient in front of $\hbar^n$ in

$$(5.1)\quad G([\frac{1}{(n+1)^i}f^i\exp(\frac{1}{\hbar}f)\frac{dx_0\dots dx_n}{d(x_0\dots x_n)}],[\frac{1}{(n+1)^j}f^j\exp(\frac{1}{\hbar}f)\frac{dx_0\dots dx_n}{d(x_0\dots x_n)}]) = \\ = \sum_{m=0}^n\frac{1}{(n+1)^{i+j}}(\int_{\Delta^+_{p_m}}f^ie^{\frac{f}{\hbar}}\frac{dx_0\dots dx_n}{d(x_0\dots x_n)}))(\int_{\Delta^-_{p_m}}f^je^{-\frac{f}{\hbar}}\frac{dx_0\dots dx_n}{d(x_0\dots x_n)}))$$

where $\{p_m\}$ is the set of critical points of f, $p_m: x_0=\dots=x_n=\zeta^m$,$\zeta=\exp\frac{2\pi\sqrt{-1}}{n+1}$; and $\Delta^+_{p_m},\Delta^-_{p_m}$ are the associated relative cycles defined above in section 3. The lowest order coefficient in (5.1) can be calculated by expanding f near critical points up to second order:

(5.2)

$$\frac{1}{(n+1)^{i+j}}\sum_{m=0}^n\int_{\Delta^+_{p_m}}f^i(p_m)\exp\big(\frac{1}{\hbar}(f(p_m)+\zeta^m\sum_{1\le k,l\le n}(1-\frac{x_k}{\zeta^m})(1-\frac{x_l}{\zeta^m}))\big)\frac{dx_1}{x_1}\dots\frac{dx_n}{x_n}]) \\ \int_{\Delta^-_{p_m}}f^j(p_m)\exp\big(\frac{1}{\hbar}(f(p_m)+\zeta^m\sum_{1\le k,l\le n}(1-\frac{x_k}{\zeta^m})(1-\frac{x_l}{\zeta^m}))\big)\frac{dx_1}{x_1}\dots\frac{dx_n}{x_n}])\sim \\ \sim\hbar^n(\sum_{m=0}^n\zeta^{m(i+j-n)})+O(\hbar^{n+1})\sim\hbar^n\delta_{i+j-n,0}+O(\hbar^{n+1})$$

Corollary 5.5. *The pairing on $T\mathcal{U}$ given by*

$$<\frac{\partial}{\partial t^i_{S_0}},\frac{\partial}{\partial t^j_{S_0}}>=\delta_{i+j,n}$$

satisfies the conditions (4.10) and (4.15).

Let us consider Taylor expansion of the constructed solution $\Phi^{S_0}(y)$ at $y=0$, where $\partial^3_{ijk}\Phi^{S_0}(y)=\sum_m\delta_{i+m,n}A^m_{jk}(y)$ and we assume that the Taylor expansion of $\Phi^{S_0}(y)$ has no terms of order ≤ 2.

Lemma 5.6. *The Taylor expansion of Φ^{S_0} has the form*

$$\Phi^{S_0} = \frac{1}{2}(\sum_{ij} y^i y^j y^0 \delta_{i+j,n}) + \sum_{m_a} \frac{\sigma(m_1, \dots, m_n)}{m_1! \dots m_n!}(y^1)^{m_1} \dots (y^n)^{m_n} \tag{5.3}$$

with

$$\sigma(m_1 + 1, m_2, \dots, m_n) = \frac{3 - n + \sum_{k=2}^{n}(k-1)m_k}{n+1}\sigma(m_1, m_2, \dots, m_n) \tag{5.4}$$

and if $\frac{3-n+\sum_{k=2}^{n}(k-1)m_k}{n+1}$ *is not integral then*

$$\sigma(m_1, m_2, \dots, m_n) = 0 \tag{5.5}$$

Proof. The form of the term which contains y^0 follows from lemma 5.3 and proposition 4.8. Equation (5.4) follows from (4.14),(4.15). Therefore expansion (5.3) can be rewritten as expansion in powers of $(y^0, e^{\frac{y^1}{n+1}} = 1 + \frac{y^1}{n+1} + \dots, y^2, \dots, y^n)$. To prove that this expansion contains only integral powers of e^{y^1} let us introduce an additional parameter. Namely, let us consider family (X_q, f) where $f = x_0 + \ldots + x_n$ and $X_q \subset \mathbb{C}^n$ is defined by equations $x_0 \cdot \ldots \cdot x_n = q$, $q \in \mathbb{C}$, $|q-1| < 1$. Identifying X_q with X via multiplication of every coordinate by $q^{-\frac{1}{n+1}}$ we see that $(X_q, f) \sim (X, q^{\frac{1}{n+1}} f)$ as deformations of (X, f). It follows from (3.3) that

$$\sum_{k=0}^{n} \alpha^k \int_{\Delta_k} \exp(q^{\frac{1}{n+1}} \frac{f}{\hbar}) \frac{dx_0 \dots dx_n}{d(x_0 \dots x_n)}] = \hbar^{-(n+1)\alpha} q^{\alpha}(1 + O(\frac{1}{\hbar}))$$

we see that $[\exp(q^{\frac{1}{n+1}} \frac{f}{\hbar}) \frac{dx_0 \dots dx_n}{d(x_0 \dots x_n)}] \in S_0 + [\Omega_0]$ and that

$$[\exp(q^{\frac{1}{n+1}} \frac{f}{\hbar}) \frac{dx_0 \dots dx_n}{d(x_0 \dots x_n)}] - [\Omega_0] = \ln q \, \alpha \hbar^{-(n+1)\alpha} \bmod \hbar^{-1} S \tag{5.6}$$

Therefore (X_q, f) corresponds to $(0, y_1, 0, \dots, 0) \in \mathcal{U}$, $q = e^{y^1}$, in the moduli space of deformations of (X, f). One can now repeat the above story starting with (X_q, f), $q \in \mathbb{C} \setminus \{0\}$ and consider element $\Psi^{S_0}(y, \hbar; q)$ representing periods of the variation of semi-infinite Hodge structures associated with moduli space of deformations of (X_q, f). The normalization condition is given by

$$\Psi^{S_0}(y, \hbar; q) - [\exp(q^{\frac{1}{n+1}} \frac{f}{\hbar}) \frac{dx_0 \dots dx_n}{d(x_0 \dots x_n)}] \in S_0$$

and the parameters $\{y^i\}$ are specified via

$$\Psi^{S_0}(y, \hbar; q) = [\exp(q^{\frac{1}{n+1}} \frac{f}{\hbar}) \frac{dx_0 \dots dx_n}{d(x_0 \dots x_n)}] + \sum_{k=0}^{n} \alpha^k \hbar^{k-1-(n+1)\alpha} y^k \ \ mod \ \ \hbar^{-1} S_0$$

Notice that this is correctly defined since $\exp(2\pi\sqrt{-1}\alpha)(S_0) = S_0$ and therefore S_0 is invariant under monodromy around $q = 0$. It follows from (5.6) that

$$\Psi^{S_0}(y_0, y_1, \dots, y_n, \hbar; q) = \Psi^{S_0}(y_0, 0, \dots, y_n, \hbar; e^{y^1} q)$$

Therefore expansion of $\partial^3 \Phi^{S_0}(y)$ can contain only integral powers of e^{y^1} which gives (5.5). $\square$

Theorem 5.7. *The Taylor expansion (5.3) of the function $\Phi^{S_0}(y)$ coincides with generating series for Gromov-Witten invariants of $\mathbb{CP}^n$:*

$$\Phi^{S_0}(y) = \mathcal{F}^{\mathbb{CP}^n}(y) \tag{5.7}$$

Proof. It follows from lemmas 5.2, 5.6, corollaries 5.4, 5.5 and theorem 3.1 from [KM]

Remark 5.8. (This remark is a part of a joint work with M.Kontsevich) It is an interesting open question whether a homological mirror symmetry conjecture can be formulated in this setting in order to explain the new mirror phenomena. A natural analog of derived category of coherent sheaves associated with (X, f) does not seem to catch all the necessary information since it is most likely that it has no non-trivial objects. A resolution of this problem might be in introducing a further generalization of $A_\infty-$categories where objects exist only in some virtual sense. On the other hand, homological mirror symmetry conjecture in opposite direction which identifies $D^bCoh(\mathbb{P}^n)$ with a version of derived Fukaya category can be formulated. A sketch of the appropriate definition of the derived Fukaya category associated with (X, f) can be found in [K2] (some technical details of the construction were checked in recent preprints math.SG0007115 & 0010032 by P.Seidel). Whether there exists some *numerical* mirror symmetry related with the opposite homological mirror symmetry conjecture (i.e. $D^bCoh(\mathbb{P}^n)$ versus a Fukaya category) is an interesting open question deserving further study.

□

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